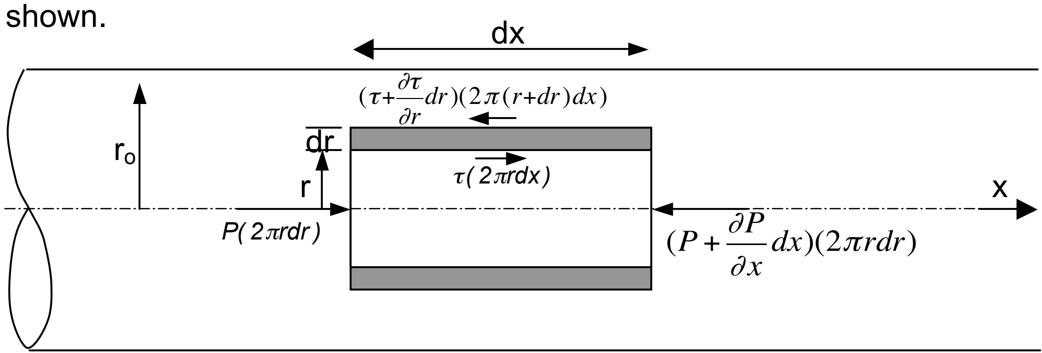
Womersley's theory for pulsating flow in straight rigid tubes

Governing equation

Consider pulsating flow in a straight rigid tube and an annular fluid element of length dx, as shown in Figure 1. Forces due to pressure and fluid shear are shown.



Assuming uniform flow (1st assumption), the balance of forces acting on the fluid element in the x-direction yields:

$\Sigma F_x = m \cdot a_x$ (Newton's 2nd law)

$$\Rightarrow P(2\pi rdr) - (P + \frac{\partial P}{\partial x}dx)(2\pi rdr) + \tau(2\pi rdx) - (\tau + \frac{\partial \tau}{\partial r}dr)(2\pi(r+dr)dx) = \rho(2\pi rdrdx)\frac{\partial v}{\partial t}$$

higher order term

$$\Rightarrow -\frac{\partial P}{\partial x}2\pi rdrdx - \tau 2\pi drdx - \frac{\partial \tau}{\partial r}2\pi rdrdx - \frac{\partial \tau}{\partial r}2\pi dr^2 dx = \rho(2\pi rdrdx)\frac{\partial v}{\partial t}$$

$$\Rightarrow -\frac{\partial P}{\partial x}r - \tau - \frac{\partial \tau}{\partial r}r = \rho r\frac{\partial v}{\partial t}$$

$$\Rightarrow -\frac{\partial P}{\partial x} - \frac{\tau}{r} - \frac{\partial \tau}{\partial r} = \rho\frac{\partial v}{\partial t} \qquad (1)$$

If we further assume that the fluid in Newtonian (2nd assumption):

$\tau = -\mu\frac{\partial v}{\partial r}$ (the minus sign comes from the imposed shear stress direction)

Substituting for τ in Eq. (1), we obtain:

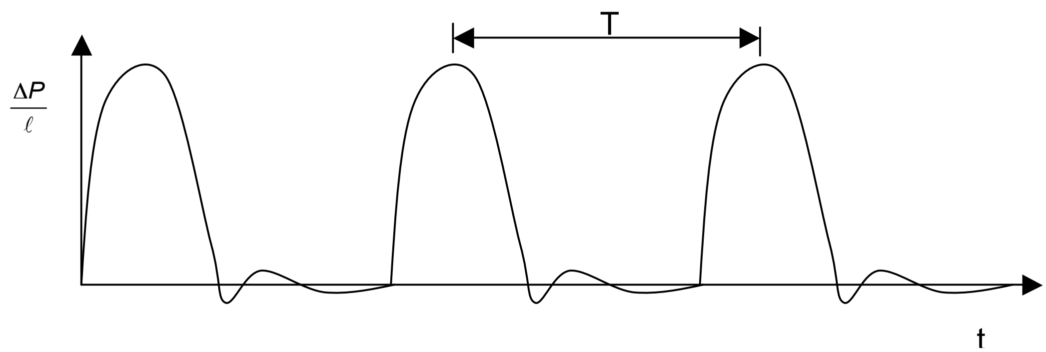
$$-\frac{\partial P}{\partial x} + \frac{\mu}{r}\frac{\partial v}{\partial r} + \mu\frac{\partial^2 v}{\partial r^2} = \rho\frac{\partial v}{\partial t}$$

$$\Rightarrow \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{\nu}\frac{\partial v}{\partial t} = \frac{1}{\mu}\frac{\partial P}{\partial x} \quad (2)$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity of the fluid. At this point we make the 3rd (logical) assumption, stating that the pressure gradient is function of time only and not a function of the radius, r: $\frac{\partial P}{\partial x} \neq f(r)$. Eq. (2) is therefore a linear partial differential equation (P.D.E.) for the time and radius dependent velocity *v(r,t).*

Solution

Equation 2 is linear, which means the general solution can be a linear superposition of other solutions. This is useful for the treatment of periodic pressure gradient functions.



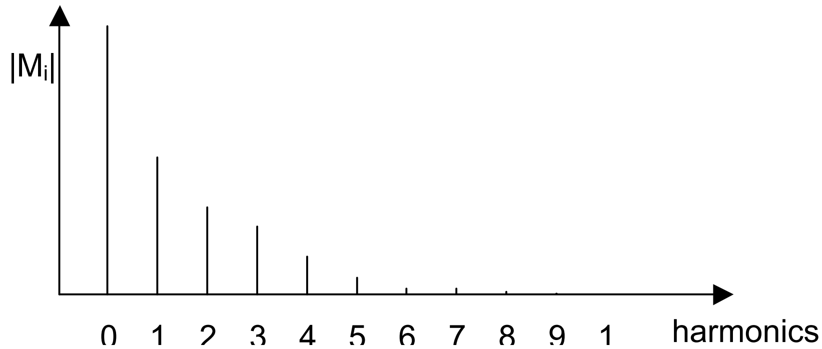
So, if we assume (4th assumption) that the pressure gradient $-\frac{\partial P}{\partial x} = \frac{\Delta P}{\ell}$ is a periodic function, with period *T*, as shown in the figure above, we can express the pressure gradient in terms of a Fourier series:

$$\frac{\Delta P}{\ell} = A_0 + A_1\cos(\omega t) + B_1\sin(\omega t) + A_2\cos(2\omega t) + B_2\sin(2\omega t) + \ldots \text{ or}$$

$$\frac{\Delta P}{\ell} = M_0 + M_1\cos(\omega t + \varphi_1) + M_2\cos(\omega t + \varphi_2) + \ldots$$

where $M_0 = A_0$, $M_i = \sqrt{A_i^2 + B_i^2}$, and $\tan\varphi_i = -\frac{B_i}{A_i}$

$\omega = \frac{2\pi}{T}$ is the circular frequency. For arterial pulses, it is in general true that 5 to 10 harmonics suffice to describe the pulse. The amplitude of higher frequency harmonics is too small and can be neglected without introducing much error.



The solution for the zero-order harmonic is obviously Poiseuille's law. Let us now consider the solution for a single harmonic pressure gradient. The general solution would be then a linear addition of Poiseuille's solution for the zero-order term plus the solution for each harmonic. For a single harmonic:

$$\frac{\Delta P}{\ell} = A\cos(\omega t) + B\sin(\omega t) = \mathrm{Re}[(A - iB)(\cos\omega t + i\sin\omega t)] = \mathrm{Re}[A^* e^{i\omega t}]$$

where $A^* = A - iB$ is a complex pressure gradient and $A^* e^{i\omega t}$ is a complex oscillatory pressure gradient whose real part is equal to the actual pressure gradient. We may now replace $\frac{\Delta P}{\ell} = -\frac{\partial P}{\partial x}$ by the complex oscillatory pressure gradient $A^* e^{i\omega t}$ in Eq. (2) to obtain:

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{\nu}\frac{\partial v}{\partial t} = -\frac{A^*}{\mu} e^{i\omega t} \qquad (3)$$

Obviously the solution to the above equation will be a complex number, whose real part will be the solution to the original linear equation (Eq. (2)). Let us now assume that the solution to Eq. (3) is given by a complex velocity $v^*(r,t)$ of the form:

$$v^*(r,t) = u(r)e^{i\omega t}$$

Hence,

$\frac{\partial v^*}{\partial t} = i\omega u e^{i\omega t}$, $\frac{\partial v^*}{\partial r} = \frac{du}{dr} e^{i\omega t}$, and $\frac{\partial^2 v^*}{\partial r^2} = \frac{d^2 u}{dr^2} e^{i\omega t}$. Substituting into Eq. (3) and dividing by $e^{i\omega t}$ we obtain:

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{i\omega}{\nu} u = -\frac{A^*}{\mu} \text{ or}$$

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{i^3\omega}{\nu}u=-\frac{A^*}{\mu} \qquad (4)$$

Eq. (4) is a linear 2[nd] order differential equation with a constant term on the right hand side. We first seek a general solution to the homogeneous equation

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{i^3\omega}{\nu}u=0 \qquad (5)$$

and then we linearly add to the solution of the homogeneous equation a particular solution satisfying Eq. (4). The homogeneous equation (4) is the know Bessel equation of order zero, which has the following homogeneous solution:

$u=C_1J_o(\lambda r)$, where $\lambda^2=\frac{i^3\omega}{\nu}$. For the particular solution, we set $u=C_2$, and substituting into Eq. (4) we obtain

$$\frac{i^3\omega C_2}{\nu}=-\frac{A^*}{\mu}\Rightarrow C_2=-\frac{A^*}{\mu}\frac{\nu}{i^3\omega}=-\frac{A^*}{\mu}\frac{\mu}{i^3\rho\omega}=-\frac{A^*}{i^3\rho\omega}$$

so that the general solution becomes

$$u(r)=C_1J_o(\lambda r)-\frac{A^*}{i^3\rho\omega} \qquad (6)$$

The constant C_1 can be evaluated by application of the non-slip boundary condition at the wall:

$$C_1=\frac{A^*}{i^3\rho\omega}\frac{1}{J_o(\lambda r_o)}$$

So the general solution (Eq. 6) becomes

$$u(r)=\frac{A^*}{i^3\rho\omega}\left[\frac{J_o(\lambda r)}{J_o(\lambda r_o)}-1\right]$$

or, using the expressions for $\lambda^2=\frac{i^3\omega}{\nu}$,

$$u(r) = \frac{A^*}{i\rho\omega}\left[1 - \frac{J_o(r\sqrt{\frac{\omega}{\nu}} \cdot i^{3/2})}{J_o(r_o\sqrt{\frac{\omega}{\nu}} \cdot i^{3/2})}\right] \qquad (7)$$

We may now define the dimensionless Womersley parameter alpha (α) as

$$\alpha = r_o\sqrt{\frac{\omega}{\nu}} = r_o\sqrt{\frac{\omega\rho}{\mu}}$$

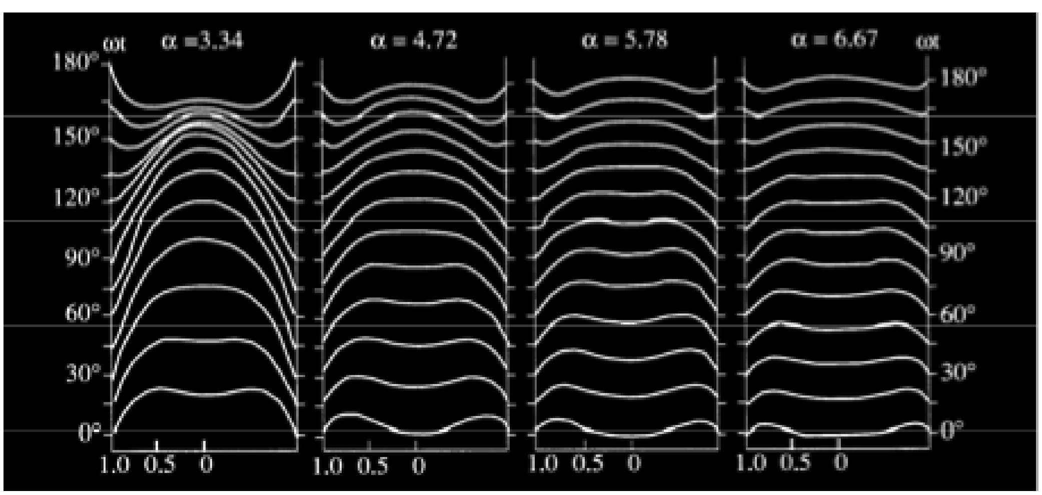
to rewrite Eq. (7) as

$$u(r) = \frac{A^*}{i\rho\omega}\left[1 - \frac{J_o(\frac{r}{r_o}\alpha \cdot i^{3/2})}{J_o(\alpha \cdot i^{3/2})}\right] \qquad (8)$$

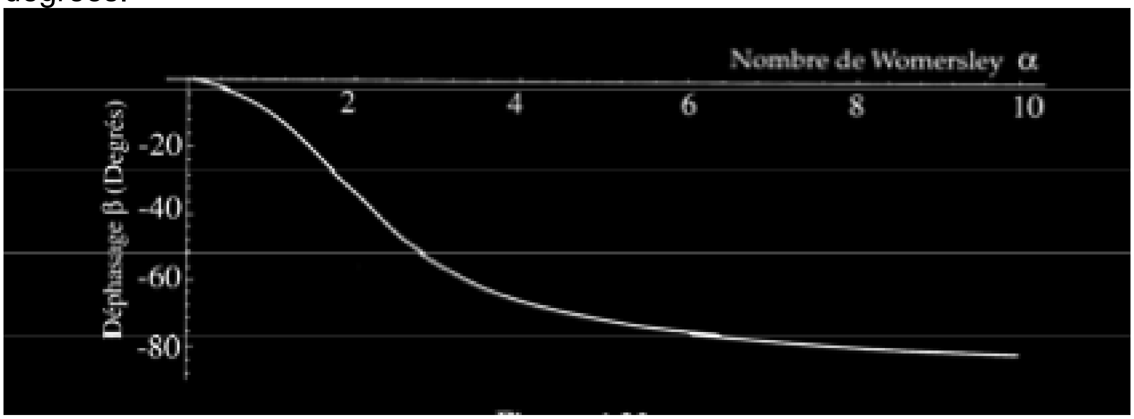
The final solution for the velocity is the real part of $v^*(r,t) = u(r)e^{i\omega t}$, so that

$$v(r,t) = \mathrm{Re}\left[\frac{A^*}{i\rho\omega}\left[1 - \frac{J_o(\frac{r}{r_o}\alpha \cdot i^{3/2})}{J_o(\alpha \cdot i^{3/2})}\right]e^{i\omega t}\right]$$

Typical velocity profiles for different Womersley parameter values are given in the figure below.



Velocity *v(r,t)* can be expressed in terms of amplitude and phase. The velocity profile shows that not all points along the radius move in phase. The phase shift, β, between the velocity *v(r,t)* and the pressure gradient $-\frac{\partial P}{\partial x}$ *(t)* is given in the following figure. For Womersley numbers approaching zero, which means for viscous-dominated flows, the phase shift is near zero. For high Womersley numbers, i.e., for inertia-dominated flows, the phase shift tends to –90 degrees, which means that velocity lags pressure gradient by 90 degrees.



Flow is obtained by integrating the velocity profile over the arterial cross-section, $Q(t) = \int_0^{r_o} v(r,t) \cdot 2\pi r \, dr$ yielding

$$Q(t) = \frac{\pi r_o^2 A^*}{i\omega\rho}\left(1 - \frac{2J_1(\alpha i^{3/2})}{\alpha i^{3/2} J_0(\alpha i^{3/2})}\right)e^{i\omega t} \qquad (9)$$

J_0 and J_1 are Bessel functions of order 0 and 1, respectively. The expression in the large parenthesis was termed [1- F_{10}] by Womersley

$$1 - F_{10} = 1 - \frac{2J_1(\alpha i^{3/2})}{\alpha i^{3/2} J_0(\alpha i^{3/2})}$$

When the real part of the pressure gradient is written as $M\cos(\omega t + \varphi)$, Eq. 9 can be written as

$$Q(t) = \frac{\pi r_o^2}{\omega\rho} M[1 - F_{10}]\sin(\omega t + \varphi)$$

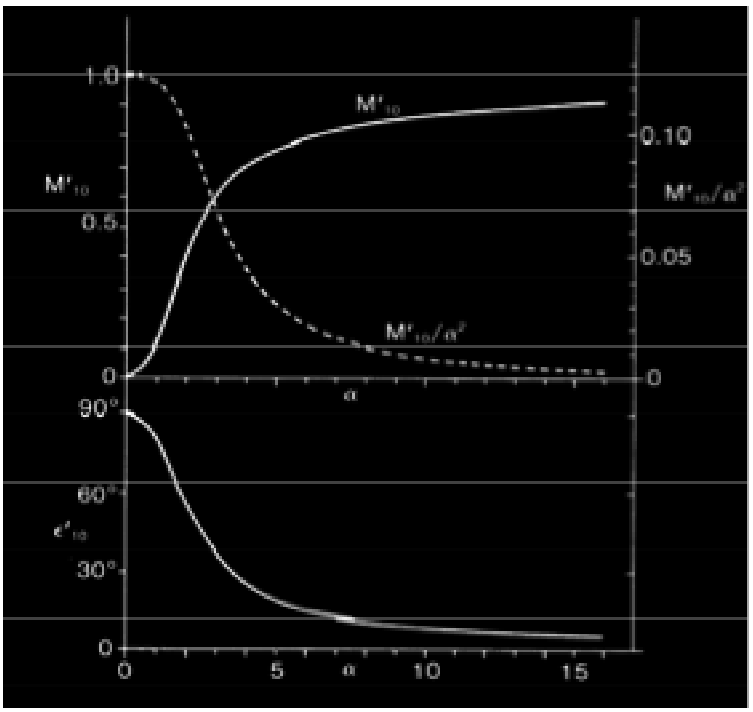
To interpret the above equation we express [1- F_{10}] in terms of its modulus (M'_{10}) and phase (ε_{10})

$$Q(t) = \frac{\pi r_o^2}{\omega \rho} M M'_{10} \sin(\omega t + \varphi + \varepsilon_{10})$$

To allow comparison with Poiseuille's equation, we substitute for the term $\alpha^2 = \frac{r_o^2 \omega \rho}{\mu}$ to obtain

$$Q(t) = \frac{\pi r_o^4}{\mu} M \frac{M'_{10}}{\alpha^2} \sin(\omega t + \varphi + \varepsilon_{10})$$

The parameters M'_{10} and ε_{10} are given in tables and graphically in the figure below.



We note that as $\alpha \to 0$, $\frac{M'_{10}}{\alpha^2} \to \frac{1}{8}$ and $\varepsilon_{10} \to 90^o$ so that

$$Q(t) = \frac{\pi r_o^4}{8\mu} M \cos(\omega t + \varphi)$$

which is effectively Poisseuille's law.

Since the heart does not generate a single harmonic but a series of harmonics, so that the flow profile *in vivo* is very complex. Experiments have shown that Womersley's theory is accurate.

Physiological and clinical relevance

Womersley's oscillatory flow theory reduces to Poiseuille's law for very low α. This means that in the periphery with small blood vessels (small *r*) and little oscillation, there is no need for the oscillatory flow theory and we can describe the pressure-flow relation with Poiseuille's law. For the very large conduit arteries, where α>10, friction does not play a significant role and the pressure-flow relation can be described with inertance alone. For α values in between, the combination of the resistance plus the inductance approximates the oscillatory pressure-flow relations.

Models of the entire arterial system have indicated that, even in the intermediately sized arteries, the oscillatory flow effects on velocity profiles are not large. The main effects are due to branching, non-uniformity and bending of the blood vessels etc. Thus for global hemodynamics, i.e., wave travel, input impedance, windkessel models etc. a segment of artery can be described by a series arrangement of a Poiseuille resistance and inductance (inertia).

The oscillatory flow theory is, however, of importance when local phenomena are studied. For instance detailed flow profiles and the calculation of shear stress at the vascular wall require the oscillatory flow theory.